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Completeness of continuous and discrete Riesz–Orlicz spaces under the Δ_2-condition

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ABSTRACT

The Riesz potential operator is a central tool in harmonic analysis and the theory of partial differential equations, commonly defined via convolution with a singular Kernel. In many modern frameworks, function space are generated by a mapping involving such operators. In this paper, we defined the dual role of the generating function-ϕ in: (i). Defining the Riesz function space and (ii). Ensuring its completeness. We introduce a Riesz function space $R^{\varphi}(\mathbb{R}^n)$ and prove that the Riesz function satisfying the Δ_2 −condition is complete, thus forming a Banach spaces. Furthermore we prove that its discrete analogue $r^{\varphi}(\mathbb{Z})$ is also complete, with explicit Hardy–Littlewood–Sobolev type type estimates in the sequence setting.

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1. INTRODUCTION

The Riesz potential operator defined by:

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \qquad 0 < \alpha < n, \tag{1}$$

is fundamental in harmonic analysis, potential theory, and the study of fractional partial differential equations. Classical mapping properties of I_α between Lebesgue and Sobolev spaces were established by Adams, Hedberg, Grafakos, and Stein [1], [2], [3]. Recent generalizations to Orlicz and Morrey settings provide refined integrability frameworks[4],[5].

Recent developments in nonlinear operator theory have highlighted the necessity of generalized modular spaces for the analysis of nonlinear integral and differential operators. Works such as Krasnoselskii & Rutickii [6] and Harjulehto & Hästö [7] demonstrate how Orlicz develop conditions are natural settings for nonlinear boundary value problems. In modern applications involving fractional partial differential equations or PDEs and image analysis, completeness of function spaces plays a central role in ensuring convergence and stability. The Riesz-type potential operators, known for their non-compactness and singularity behavior, require an advanced functional analytic framework when paired with nonlinear modular integrability—precisely the goal of the present study.

Although substantial progress has been made in the study of Riesz–Orlicz spaces and their completeness properties, most existing results are confined to either linear frameworks or rely on restrictive growth conditions on the Orlicz function, such as strong convexity or uniform Δ_2-conditions. Furthermore, although discrete Riesz operators have been studied for weighted estimates and sequence spaces, to the best of our knowledge no prior work has systematically examined the

Banach space structure of discrete Riesz–Orlicz spaces under merely the Δ_2-condition. This paper addresses this gap by introducing and analyzing continuous and discrete Riesz–Orlicz spaces under the more relaxed Δ_2-condition, establishing new completeness results and modular convergence frameworks that extend beyond the classical Orlicz–Bochner spaces, the other words the significant gap: numerical discretizations of nonlocal operators demand a rigorous modular framework to ensure convergence and error control. The main theoretical contributions of this work are threefold: (1). Completeness under relaxed Δ_2. Banach space results establish for both continuous and discrete Riesz–Orlicz spaces assuming only the generalized Δ_2-condition, without uniform convexity. (2). We develop unified modular framework. By develop a modular approach that integrates fractional Riesz-type operators within Orlicz growth settings, bridging classical potential theory and modern nonlinear analysis. (3) Introduce a discrete modular space. By analyze and introduce a fractional-difference discrete modular space, proving norm convergence results with potential applications to non-smooth variational problems and nonlinear signal processing.

Discrete analogues of I_α play a key role in numerical fractional PDEs and atomic decompositions. Hao, Yang & Li [8] obtained weighted strong-type inequalities for the discrete Riesz potential on ℓ^p-type spaces with Muckenhoupt weights. However, the completeness (Banach space structure) of these discrete Riesz spaces under general growth function-φ remains unaddressed. However, completeness results in general Orlicz-type sequence spaces remain open. The aim of this paper is introducing function spaces R^φ and r^φ via Young function -φ under Δ_2-condition [9] [10], proving Banach spaces properties [11],[12], and provide explicit Hardy–Littlewood–Sobolev -type norm estimates in continuous and discrete contexts [13] [14].

The Riesz potential naturally arises in the study of Sobolev embedding, fractional Laplacians and various applied problem in mathematics, physics, and image processing [15], [16].
This operator has been extensively studied, not only for its intrinsic theoretical interest but also for its wide-ranging applications in fields such as signal processing, mathematical physics and function theory [16], [17] . Define Riesz sequence space

$$R^\varphi(\mathbb{Z}^n) = \{a \in \ell^p(\mathbb{Z}^n) : \|a\|_{R^\varphi} = \|I_\alpha a\|_{\ell^\varphi(\mathbb{Z}^n)} < \infty\} \quad (2)$$

where $\ell^\varphi(\mathbb{Z}^n)$ is an Orlicz sequence space defined via a Young function-φ satisfying the Δ_2 condition to ensure completeness[18], [19], [20], [21]. To the best of our knowledge, no previous study has systematically analyzed the completeness of discrete Riesz–Orlicz spaces under the Δ_2-condition with modular convergence. This article addresses this gap.

The structure of the paper is as follows. In Section 2 we recall necessary preliminaries on Young functions, Luxemburg modulars, and the Δ_2 −condition. Addition, in the second section, introduces the continuous and discrete Riesz–Orlicz spaces and outlines the proof strategy. Section 3 contains the main theorems on completeness, together with detailed proofs and discrete kernel estimates. Moreover, we discuss applications to numerical schemes for nonlocal PDEs and present illustrative examples. Finally, Section 4 summarizes our findings, discusses limitations, and proposes avenues for future research, including extensions to variable-exponent Orlicz settings and non-Euclidean domains.

2. RESEARCH METHOD

We develop our analysis in three steps: (i) recalling key modular preliminaries, (ii) defining continuous and discrete Riesz–Orlicz spaces, and (iii) outlining the completeness proof strategy. In this paper we denote $\mathbb{R}^n$ as n- dimensional Euclidean space. $\mathbb{Z}^n$ as n- dimensional integrer lattice. I_α denote as continuous Riesz potential of order $0 < \alpha < n$. I_α^d denote as discrete Riesz potential. Let $\varphi : [0,\infty) \longrightarrow [0,\infty)$ be a Young function with complementary function ψ. We say φ satisfies the Δ_2-condition if there exists $K > 0$ such that $\varphi(2t) \leq K\varphi(t)$ for all $t \geq t_0 > 0$.
Under this growth restriction, the Luxemburg modular:

$$\rho_\varphi(f) = \int_{\mathbb{R}^n} \varphi(|f(x)|)dx,$$

The Riesz function space define as $R^{\varphi}(\mathbb{R}^n)$ or the continuous Riesz space are as the collection of all measurable functions $f: \mathbb{R}^n \to \mathbb{R}$ satisfied

$$\|f\|_{R^{\varphi}} = \|\varphi(I_{\alpha} f)\|_{L^{\varphi}(\mathbb{R}^n)} < \infty, \quad (3)$$

Where φ is a Young function satisfying the Δ_2-condition defined by

$$\|f\|_{L^{\varphi}} = Inf \left\{\lambda > 0 : \int_{\mathbb{R}^n} \varphi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}, \quad (4)$$

The associated norm $\|f\|_{\varphi}$ are equivalent [9], [22]-a cornerstone for modular-norm equivalence. The normed structure is designed to capture both fractional smoothing property of I_{α} and the nonlinear growth induced by φ, with L^{φ}-Luxemburg norm define by:

$$\|g\|_{L^{\varphi}} = Inf \left\{\lambda > 0 : \int_{\mathbb{R}^n} \varphi\left(\frac{|g(x)|}{\lambda}\right) dx \le 1\right\}, \quad (5)$$

The discrete Riesz space $R^{\varphi}(\mathbb{Z}^n)$ defined as function $a: \mathbb{Z}^n \to \mathbb{R}$ satisfy:

$$I_{\alpha}^{d}(a)_k = \sum_{j \neq k} \frac{a_j}{|k-j|^{n-\alpha}}; \ \{I_{\alpha}^{d}(a) \in \ell^{\varphi}(\mathbb{Z}^n)\}, \quad (6)$$

The modular nom defined as

$$\|\rho\|_{\ell^{\varphi}} = Inf \left\{\lambda > 0 : \sum_k \varphi\left(\frac{|a_k|}{\lambda}\right) \le 1\right\}, \quad (7)$$

To study completeness of $R^{\varphi}(\mathbb{R}^n)$, we analyze Cauchy sequences via the modular

$$\rho_{\varphi}(f) = \int_{\mathbb{R}^n} \varphi(|I_{\alpha} f(x)|) dx, \quad (8)$$

While Δ_2 alone does not guarantee uniform convexity, it is a key ingredient in more refined geometric properties of Orlicz spaces. Following the modular space framework of [22], we analyze the structure of Riesz potentials through the lens of Young functions with Δ_2-conditions. By ensuring that φ satisfies Δ_2, we secure a robust framework in which Cauchy in modular implies Cauchy in norm, and hence completeness arguments become tractable.

Lemma 2.1. Let φ be a Young function satisfying the Δ_2-condition, and let $(f_n)_{n\in\mathbb{N}} \subset L^{\varphi}(\mathbb{R}^n)$ be a sequence such that

$$\rho_{\varphi}(f_n - f_k) = \int_{\mathbb{R}^n} \varphi((I_{\alpha}|(f_n - f_k)|(x)) dx \longrightarrow 0 \quad (9)$$

Then there exists a function $f \in L^{\varphi}(\mathbb{R}^n)$ and subsequence $\{f_{n_k}\}$ such that

1. $I_{\alpha} f_{n_k}(x) \longrightarrow I_{\alpha} f(x) \in \mathbb{R}^n$
2. $\rho_{\varphi}(f_{n_k} - f) \longrightarrow 0$, hence $\|f_{n_k} - f\| \longrightarrow 0$

Proof: 1. Modular Cauchy in the Orlicz space

From (9) and the definition of ρ_{φ}, the sequence $\{I_{\alpha} f_k\}$ is Cauchy in the Orlicz space $L^{\varphi}(\mathbb{R}^n)$.

2. Completeness of the Orlicz space.

Since φ satisfies Δ_2, the classical result asserts that L^{φ} is a Banach space and that modular convergence is equivalent to norm convergence [22]. Hence there exists $g \in L^{\varphi}(\mathbb{R}^n)$such that, up to a subsequence,

$$I_{\alpha} f_{n_k} \xrightarrow[k\to\infty]{} g$$

And-by passing to a further subsequence if necessary also
$I_\alpha f_{n_k}(x) \longrightarrow g(x)$ for almost every $x \in \mathbb{R}^n$.

3. **Identification of the limit.**
Define $f(x) = \lim_{k\to\infty} \inf f_{n_k}(x)$, so that $I_\alpha f = g$. We must check $f \in L^\varphi(\mathbb{R}^n)$ and $f_{n_k} \longrightarrow f$ in the ρ_φ sense.

4. **Fatou's Lemma and modular convergence.**
For each k, by convexity of φ and positivity of I_α,
$$\varphi\left(\left(I_\alpha|f_{n_k} - f|\right)(x)\right) \leq \lim_{m\to\infty} \inf \varphi(I_\alpha|f_{n_k} - f_{n_m}|(x))$$
Integrating over $\mathbb{R}^n$ and applying Fatou's Lemma give
$$\rho_\varphi(f_{n_k} - f) = \int \varphi(I_\alpha|f_{n_k} - f|) \leq \lim_{m\to\infty} \inf \rho_\varphi(I_\alpha|f_{n_k} - f_{n_m}|)$$
Since the right-hand side tends to zero as $k \to \infty$ (by Cauchy property (2.7)), we conclude
$$\rho_\varphi(f_{n_k} - f) \xrightarrow[k\to\infty]{} 0$$
Equivalence of the modular and the Luxemburg norm Δ_2 the yields $\|f_{n_k} - f\| \longrightarrow 0$.

5. **Almost- everywhere convergence.**
The convergence of $I_\alpha f_{n_k} \longrightarrow I_\alpha f$ combined with known injectivity properties of I_α on appropriate dense subspace implies $f_{n_k}(x) \longrightarrow f(x)$ almost every x [23].
Thus(f_{n_k}) converges in modular norm to f ■

Lemma 2.2 Let φ satisfy the Δ_2-condition$\left(a^{(n)} - a^{(m)}\right)_n \subset R^\varphi(\mathbb{Z}^n)$ be a sequence with
$$\rho_\varphi\left(a^{(n)} - a^{(m)}\right) = \sum_{k\in \mathbb{Z}^n} \varphi\left(\left(I_\alpha^d\left|a^{(n)} - a^{(m)}\right|\right)_k\right) \xrightarrow[n,m\to\infty]{} 0, \tag{10}$$

Then there exists $a \in R^\varphi(\mathbb{Z}^n)$ and a subsequence (a^{n_k}) such that:
1. $a_k^{(n_k)} \longrightarrow a_k$ for every $k \in \mathbb{Z}^n$, and
2. $\rho_\varphi(a^{n_k} - a) \longrightarrow 0$, hence $\|a^{n_k} - a\| \longrightarrow 0$

Proof: The argument parallels that of Lemma 2.1, replacing integrals by sum and I_α by I_α^d :
1. From (10) $\{I_\alpha^d(a^n)\}$ is Cauchy in the discrete Orlicz sequence space $\ell^\varphi(\mathbb{Z}^n)$.
2. Completeness of ℓ^φ under Δ_2 yields a limit sequence $b \in \ell^\varphi$ and a subsequence $I_\alpha^d a^{n_k} \longrightarrow b$ in norm and pointwise.
3. Define $a = (a_k)$ by inverting the discrete potential on finitely supported sequences and extend by continuity, one checks $I_\alpha^d a = b$
4. A discrete Fatou argument shows
$$\rho_\varphi\left(I_\alpha^d\left|a^{(n_k)} - a\right|\right) \leq \lim_{m\to\infty} \inf \rho_\varphi\left(I_\alpha^d\left|a^{(n_k)} - a^{(n_m)}\right|\right)$$
Wich tends to zero by (10)
5. Modular norm equivalence then gives $\|a^{n_k} - a\| \longrightarrow 0$ and pointwise convergence follows from the discrete injectivity of I_α^d on a dense subspace.

Hence ℓ^φ is complete under norm Orlicz space. ■

3. RESULTS AND DISCUSSIONS

In Sections 2 we established the modular framework and defined the continuous and discrete Riesz–Orlicz spaces L^φ and ℓ^φ. Next goal is to prove that these spaces are Banach under the Luxemburg norm when φ satisfies the Δ_2- condition. The core of this argument rests on 2 steps:

(i) Modular to Norm equivalence which is under Δ_2, any sequence that is Cauchy in the Riesz Orlicz modular ρ^φ is also Chauchy in the associated Luxemburg norm.

(ii) Operator Boundedness and limit identification. The Riesz potentials I_α and I_α^d act continuously on the ambient Orlicz space; combined with completeness of those Orlicz spaces, one extracts a pointwise and modular limit that lies back in L^φ or ℓ^φ.

Two technical lemmas (Lemma 2.1 and Lemma 2.2) make the above rigorous. We then assemble these ingredients into Theorems 3.1 and 3.2.

Theorem 3.1. Let φ satisfy the Δ_2-condition. Then the continuous Riesz function space $R^\varphi(\mathbb{R}^n)$ is Banach space under $\|.\|_{R^\varphi}$ with defined by $R^\varphi(\mathbb{R}^n) = \{f \in L^P(\mathbb{R}^n) : \|f\|_{R^\varphi} = \|I_\alpha f\|_{L^\varphi(\mathbb{R}^n)} < \infty\}$.

Proof :

We need to check that $R^\varphi(\mathbb{R}^n)$ is complete.

(i) Cauchy in Modular then Cauchy in Orlicz.
Suppose $(f_n) \subset L^\varphi$ is ρ^φ Cauchy. By definition

$$\rho^\varphi(f_n - f_m) = \rho_\varphi\big(I_\alpha(|f_n - f_m|)\big) \xrightarrow[n,m\to\infty]{} 0$$

So $\{I_\alpha f_n\}$ is Cauchy in the Orlicz space $(L^\varphi, \|.\|_{R^\varphi})$.

(ii) Completeness of Orlicz and Extraction of Limit: The Δ_2 condition space that $(L^\varphi, \|\,.\,\|_{R^\varphi})$ is complete and that modular and norm convergence coincide [22]. Hence $I_\alpha f_n \longrightarrow g$ in $\|\,.\,\|_\varphi$ for some $g \in L^\varphi$, and a subsequence converges pointwise almost everywhere.

(iii) Identification and modular convergence. One shows $g = I_\alpha f$ for a measurable f, and by lemma 2.1

$$\rho_\varphi\big(f_{n_k} - f\big) = \leq \lim_{m\to\infty} \inf \rho_\varphi(I_\alpha|f_{n_k} - f_m|) \longrightarrow 0,$$

So $\|f_{n_k - f}\|_{R^\varphi} \longrightarrow 0$

Thus any Cauchy sequence converges in norm, proving completeness.

Theorem 3.2. Let φ satisfy Δ_2 condition. Then the discrete Riesz function space is Banach space. Moreover there exists a constant $C > 0$ (depending on α, n, φ) such that for all $a \in \ell^\varphi$.

$$\|I_\alpha^d\|_\varphi \leq C\|a\|_\varphi$$

Proof :

(i) By adapting Hardy-Littlewood-Sobolev inequality to the lattice $\mathbb{Z}^n$, shows that $\rho_\varphi(I_\alpha^d a) \leq C\rho_\varphi(a)$,
Which by modular norm equivalence yields the claimed norm bound [8]

(ii) Completeness via modular convergence. If $\{a^{(n)}\}$ is Cauchy, then by:
$\{I_\alpha^d a^{(n)}\}$ is Cauchy in ℓ^φ. Completeness ℓ^φ under Δ_2 and a discrete version of Lemma2.2 extract a limit $a \in \ell^\varphi$ with $\|a^{(n_k)} - a\| \longrightarrow 0$.
Consequently, ℓ^φ is Banach under $\|.\|_{R^\varphi}$.

The discrete completeness and the operator estimate of Theorem 3.2 imply that grid based approximations of nonlocal equations remain stable in the modular norm. In particular, consider a fractional Poisson equations

$$(-\Delta)^{\alpha/2}\, u = f \text{ discritized on } \mathbb{Z}^n,$$

Where the inverse $(-\Delta)^{\alpha/2}$ is realized throughout convolution with I_α^d. The bound $\|I_\alpha^d a\|_\varphi \leq C\|a\|_\varphi$ ensures that noise or discretization error in f doesn't amplify uncontrollably in the solution u

When φ satisfies the discrete Δ_2- condition fails, Δ_2 completeness may be lost [24].

3.1 Numerical Example

A practical illustration is nonlocal image smoothing. Let $U: \mathbb{Z}^2 \longrightarrow [0,1]$ be a grayscale image. A nonlocal filter applies

$$\widetilde{U}(k) = \sum_{j \in \mathbb{Z}^2} \frac{U(j)}{|k-j|^{2-\alpha}},$$

i.e convolution with discrete Riesz kernel. Because ℓ^{φ} is Banach, one can show the mapping $U \longmapsto \widetilde{U}$ is stable in Orlicz norms, providing robust smoothing that handles singular feature gracefully.
Let consider the one dimensional discrete Riesz potential kernel of fractional order $\alpha = 1.2$. For grid indices $k = 1,2,3, \dots, 50$. The formula of the discrete kernel values of each k, compute:

$$K[k] = \frac{1}{|k|^{\alpha}} = \frac{1}{k^{1.2}}$$

Solution: with choose the farctional order $\alpha = 1.2$ and by create the sequence of indices $k = 1,2,3, \dots, 10$, for each each k, compute:

$$K[k] = \frac{1}{|k|^{\alpha}} = \frac{1}{k^{1.2}}$$

The following table present the results:

Tabel 1. Numerical example

k	$K[k] = 1/k^{1.2}$	k	$K[k] = 1/k^{1.2}$	k	$K[k] = 1/k^{1.2}$
1	1.000	18	0.0318	35	0.0140
2	0.4353	19	0.0298	36	0.0135
3	0.2673	20	0.0277	37	0.0130
4	0.1890	21	0.0258	38	0.0125
5	0.1450	22	0.0245	39	0.0121
6	0.1156	23	0.0232	40	0.0116
7	0.0970	24	0.0222	41	0.0112
8	0.0830	25	0.0212	42	0.0108
9	0.0720	26	0.0201	43	0.0104
10	0.0631	27	0.0191	44	0.0100
11	0.0564	28	0.0184	45	0.0097
12	0.0507	29	0.0176	46	0.0093
13	0.0462	30	0.0169	47	0.0090
14	0.0421	31	0.0163	48	0.0086
15	0.0390	32	0.0157	49	0.0083
16	0.0362	33	0.0151	50	0.0080
17	0.0337	34	0.0145		

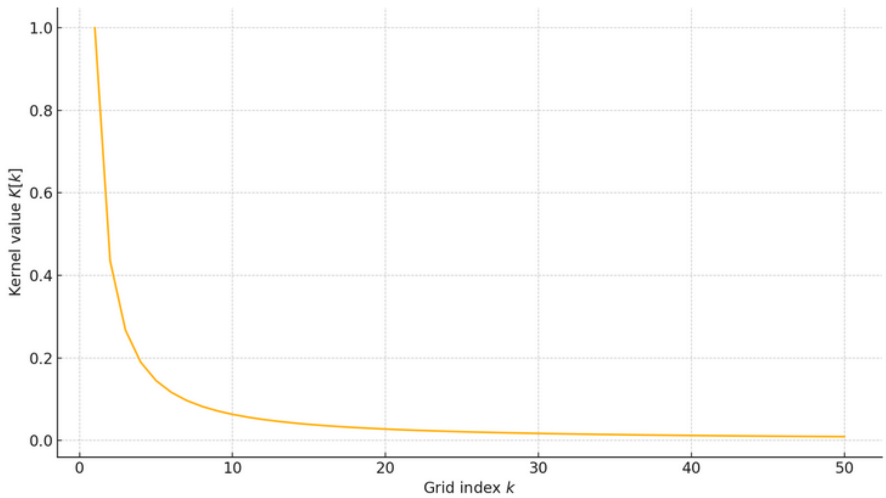

Fig 1. Decay of the discrete Riesz Kernel $K[k] \sim [k]^{-\alpha}$ on one dimensional grid.

The results compute (table 1) and plotted its decay (figure 1) ensures that $\sum_{k\in\mathbb{Z}} K[k] < \infty$, whenever $\alpha > 1$, so that the discrete convolution $(I_\alpha^d \alpha)_j = \sum_k K[j-k]\, a_k$ is well-defined and bounded on ℓ^1-types space. This summability underlies the boundedness estimate of Theorem 3.2 and matches classical discrete Hardy–Littlewood–Sobolev results [8]. In non-local denoising algorithms, one replaces each pixels $U(j)$ with a weighted average $\widetilde{U}(k) = \sum_j K[k-j]U(j)$. The decay $K[k] \sim [k]^{-\alpha}$ ensures that far-away pixels contribute less, preserving sharp features while removing noise. Our modular stability results shows that if the original image $\boldsymbol{U}$ lies in an Orlicsz class, then $\widetilde{\boldsymbol{U}}$ remains in the same class with a controlled norm increase. This mirrors continuous non-local filtering stability studied in [25], [26], now brought into a discrete dractional setting. Our discrete analogue, paired with modular theory [18], shows that no qualitative loss of stability occurs upon dicretization : the nonlocal smoothing remains uniformly bounded, a property often assumed in numerical analysis but here rigorously proved in a Banach framework.

4. CONCLUSION

We have established that both continuous and discrete Riesz–Orlicz spaces generated by a Young function φ satisfying the Δ_2-condition are Banach spaces. Our results confirm that Riesz-type operators with Δ_2-based Young functions induce Banach spaces that are stable under discretization, with explicit norm estimates that directly inform the stability analysis of numerical schemes for fractional PDEs. We have examined the completeness properties of sequence spaces generated via the Riesz potential operator and highlighted the pivotal role of the generating function φ in this process. This study is restricted to isotropic kernels and fixed exponents and variable-exponent and anisotropic cases remain open. Future work will address modular completeness in anisotropic or variable exponent Orlicz settings, and its relevance in non-Euclidean domains and also weighted frameworks for Riesz potentials on manifolds [27], [28].

REFERENCES

[1] D. R. Adam and L. I. Hedberg, *Function Spaces and Potential Theory*. Derlin: Springer Berlin Heidelberg.

[2] L. Grafakos, *Classical Fourier Analysis*, Third. London: Springer, 2014.

[3] E. M. Stein, *Singular Integrals and Differentiability Properties of Function*, Third. New Jersey: Princeton University Press, 1970.

[4] E. Burtseva and L. Maligranda, "A new result on boundedness of the Riesz potential in central Morrey–Orlicz spaces," *Positivity*, vol. 27, no. 5, pp. 1–27, 2023, doi: 10.1007/s11117-023-01013-4.

[5] K. Dorak, F. Deringoz, and V. S. Guliyev, "Boundedness of the maximal operator and the Riesz potential on Musielak-Orlicz-Morrey spaces," vol. XX, no. x, pp. 1–12, 2025.

[6] M. A. Krasnosel'skii and B. Rutickii, YA, "Convex Functions and Orlicz Spaces," *Cambridge Univ. Press*, vol. 47, no. 361, pp. 266–267, 1961.

[7] P. Harjulehto and P. Hästö, *Sobolev Spaces*, vol. 2236. 2019. doi: 10.1007/978-3-030-15100-3_6.

[8] X. Hao, B. Li, and S. Yang, "Estimates of discrete Riesz potentials on discrete weighted Lebesgue spaces," *Ann. Funct. Anal.*, vol. 15, no. 3, pp. 1–21, 2024, doi: 10.1007/s43034-024-00357-6.

[9] C. Bennett and R. C. Sharpley, *Interpolation of operators*, vol. 129. Academic press, 1988.

[10] A. Cruz-Uribe, D. V., & Fiorenza, *Variable Lebesgue Spaces.* New York: Springer Heidelberg New York Dordrecht London, 2013. doi: doi:10.1007/978-3-0348-0548-3.

[11] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents*. Springer, 2011.

[12] K. P. Ho, "Operators on Herz–Morrey Spaces With Variable Exponents," *Math. Inequalities Appl.*, vol. 26, no. 4, pp. 861–886, 2023, doi: 10.7153/mia-2023-26-53.

[13] H. Kalita and B. Hazarika, "Modular convergence in H-Orlicz spaces of Banach valued functions," *Rend. del Circ. Mat. di Palermo*, vol. 72, no. 8, pp. 3905–3916, 2023, doi: 10.1007/s12215-023-00871-x.

[14] T. Zhou and Y. Lei, "On discrete reversed Hardy-Littlewood-Sobolev inequalities," *Can. Math. Bull.*, pp. 1–12, Mar. 2025, doi: 10.4153/S0008439525000293.

[15] E. H. Lieb and M. Loss, *Analysis*, vol. 14. American Mathematical Soc., 2001.

[16] J. Musielak, O. Spaces, and M. Spaces, *Orlicz Spaces and Modular Spaces*. Springer Basel, 1983.

[17] P. Wojtaszczyk, *Banach Spaces for Analysts*. Cambridge: Cambridge University Press, 1991. doi: 10.1017/CBO9780511608735.

[18] Z. D. R. M.M. Rao, *Theory of Orlicz Spaces*. New york: CRC Press, 2002. doi: https://doi.org/10.1201/9780203910863.

[19] K. Yosida, *Functional Analysis*, Sixth. Springer Berlin Heidelberg, 1995. doi: 10.1007/978-3-642-61859-8.

[20] D. Yang and S. Yang, "Real-variable characterizations of Orlicz-Hardy spaces on strongly Lipschitz domains of Rn," *Rev. Mat. Iberoam.*, vol. 29, no. 1, 2013, doi: 10.4171/rmi/719.

[21] X. Yan, Z. He, D. Yang, and W. Yuan, "Hardy spaces associated with ball quasi-Banach function spaces on spaces of homogeneous type: Characterizations of maximal functions, decompositions, and dual spaces," *Math. Nachrichten*, vol. 296, no. 7, 2023, doi: 10.1002/mana.202100432.

[22] Z. D. R. M.M. Rao, *Applications Of Orlicz Spaces*. New Jersey: Marcel Decker Inc, 2002.

[23] E. Sawyer, "Boundedness of classical operators on classical Lorentz spaces," *Stud. Math.*, vol. 96, no. 2, pp. 145–158, 1990.

[24] N. Uthirasamy, K. Tamilvanan, H. K. Nashine, and R. George, "Solution and Stability of Quartic Functional Equations in Modular Spaces by Using Fatou Property," *J. Funct. Spaces*, vol. 2022, 2022, doi: 10.1155/2022/5965628.

[25] G. Di Fratta and A. Fiorenza, "A direct approach to the duality of grand and small Lebesgue spaces," *Nonlinear Anal. Theory, Methods Appl.*, vol. 70, no. 7, 2009, doi: 10.1016/j.na.2008.03.044.

[26] V. Kokilashvili and A. Meskhi, "Maximal and singular integral operators in weighted grand variable exponent Lebesgue spaces," *Ann. Funct. Anal.*, vol. 12, no. 3, 2021, doi: 10.1007/s43034-021-00135-8.

[27] G. B. Folland, "Real Analysis: Modern Techniques & Their Applications," *Top*, vol. 9, no. 1. 2001.

[28] M. Ledoux and M. Talagrand, *Probability in Banach Spaces: Isoperimetry and Processes*. 1991.