International Journal of Basic and Applied Science 13 (3) (2024) 146-156

Published by: IOCSCIENCE

International Journal of Basic and Applied Science

Journal homepage: www.ijobas.pelnus.ac.id

Advancing optimization algorithms with fixed point theory in generalized metric vector spaces

Desi Vinsensia [1], Yulia Utami [2], Benavides Khan Awawdeh [3], and Nocedals Bertesh Bausch [4]

[1,2] Bisnis Digital, STMIK Pelita Nusantara, Indonesia

[3,4] Faculty Affairs Educational Technologis, City University of New York, US

Article Info

Article history:

Received May 09, 2024
Revised Jun 20, 2024
Accepted Dec 22, 2024

Keywords:

Adaptive Iterative Algorithm;
Fixed Point Theory;
Generalized Metric Vector Spaces;
Machine Learning;
Non-Convex Optimization.

ABSTRACT

This research develops and evaluates an adaptive parameter-based fixed point iterative algorithm within generalized metric vector spaces to improve stability and convergence speed in optimization problems. The study extends fixed point theory beyond classical metric spaces by incorporating a more flexible structure that accommodates non-Euclidean systems, commonly found in machine learning, data analysis, and dynamic systems optimization. The proposed adaptive fixed point algorithm modifies the conventional iterative method: $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(x_n)$ where the adaptive parameter α_n dynamically adjusts based on the previous iterations: $\alpha_n = \frac{1}{1+\beta G(x_n, x_{n-1}, x_{n-2})}$ with $\beta > 0$ as a control constant. A numerical case study demonstrates the algorithm's effectiveness, comparing it with the classical Banach Fixed Point Theorem. Results show that the adaptive method requires fewer iterations to achieve convergence while maintaining higher stability, significantly outperforming the standard approach. The findings suggest that incorporating adaptive parameters in fixed point iterations enhances computational efficiency, particularly in non-convex optimization and deep learning training models. Future research will explore the algorithm's robustness in high-dimensional spaces, its integration with hybrid optimization techniques, and applications in uncertain and noisy environments.

Corresponding Author:

Desi Vinsensia,
Bisnis Digital,
STMIK Pelita Nusantara,
Jl. Iskandar Muda No. 1 Medan, Sumatera Utara 20154, Indonesia
Email: desivinsensia87@gmail.com

1. INTRODUCTION

Fixed point theory is a branch of mathematics that studies the conditions under which a function or mapping has a fixed point, which is a value of x^* that satisfies $f(x^*) = x^*$ [1], [2], [3]. In the context of optimization, fixed point theory is very useful in analyzing the convergence of iterative algorithms and ensuring the existence of exact solutions to complex problems[4][5]. One of the fundamental results in fixed point theory is the Banach Fixed Point Theorem, which states that if a function f is a contractive mapping on a complete metric space, then it has a unique fixed point and the iterative process will always converge to that fixed point[6].

A mapping is said to be contractive if there exists a constant $0 \leq k < 1$ such that it holds[7]:

$$d\big(f(x)f(y)\big) \leq kd(x,y), \qquad \forall x,y \in X \tag{1}$$

where $d(x,y)$ is a metric that measures the distance between two points x and y. This theorem is very important in optimization because it guarantees that iteration-based algorithms, such as the Newton-

Raphson method and gradient descent, will always lead to a stable solution, provided that the mapping used satisfies the contractive property[8]. For example, in an iterative method to find the roots of a function, we can use the scheme[9], [10], [11]:

$$x_{n+1} = f(x_n) \tag{2}$$

If f is a contractive mapping, then this iteration will converge to a fixed point x^*, which is the optimal solution of the optimization problem.

In addition to contractive mappings, there is the concept of quasicontractive mappings, which is a weaker form but still useful in convergence analysis. A mapping is said to be quasicontractive if for all x, y, there exists a constant$k \in [0,1)$ such that[12], [13], [14], [15]:

$$d\big(f(x), f(y)\big) \leq d(x, y) - kd\big(x, f(x)\big) \tag{3}$$

Quasicontractive mapping often arises in more complex optimization problems, such as projection in metric vector space or nonsmooth convex optimization, where convergence is not always exponential as in contractive mapping but still guarantees the existence of fixed points[15].

The application of fixed point theory in optimization is very wide, including in the Proximal Point method, which is used in convex optimization to solve problems[16], [17], [18], [19]:

$$x_{\lambda+1} = \text{agr} \min_x \left(f(x) + \frac{1}{2\lambda} \|x - x_\lambda\|^2 \right) \tag{4}$$

In this scheme, proximal operators are often quasicontractive mapping that ensures iterations stay within the desired domain and converge to the optimal solution. Thus, fixed-point theory, particularly for contractive and quasicontractive mapping, has proven to be a very powerful tool in optimization analysis, providing a guarantee of convergence and precise solutions in a wide range of complex problems, from machine learning to network and dynamic systems optimization[19].

This research aims to extend fixed point theory into generalized metric vector spaces, which is a development of classical metric spaces with more flexible additional structures. Generalized metric vector spaces allow for a wider definition of distance, so they can cover a wide range of non-Euclidean systems that often appear in optimizations in machine learning and data analysis. By developing a new fixed-point theorem in this space, we can guarantee the convergence of solutions in a more general environment. Mathematically, if (X, d) is a generalized metric vector space and $T: X \to X$ is a contractive mapping with factor k, then the existence of a fixed point x^* is guaranteed by the condition[12], [13], [20], [21]:

$$d\big(T(x), T(y)\big) \leq kd(x, y), \qquad \forall x, y \in X, \qquad 0 \leq k < 1 \tag{5}$$

In addition, this study will develop an optimization algorithm based on fixed point theory by adapting an iterative method that utilizes contractive and quasicontractive mapping properties. One of the iterative schemes that can be used is[16], [17], [18], [22], [23]:

$$x_{n+1} = T(x_n) \tag{6}$$

where T is a proximal operator that ensures convergence to the optimal solution. This approach can be applied in various methods, including convex optimization and machine learning, with the advantage of better stability over conventional approaches.

As part of the empirical study, this study will also examine the effectiveness of this method compared to classical optimization algorithms such as gradient descent and Newton's method. For example, in gradient descent, the parameters are updated based on[11], [24], [25], [26]:

$$\theta_{t+1} = \theta_t - \eta \nabla J(\theta_t), \tag{7}$$

whereas in Newton's method, the update is carried out by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{8}$$

Both methods have challenges, such as sensitivity to the selection of learning rates in gradient descent and Hessian matrix calculations in Newton's method. Using a fixed point theory approach, the algorithm developed is expected to have a more stable and faster convergence, especially in complex and high-dimensional optimization problems. Therefore, this research will make a significant contribution in developing fixed-point theory into a more general metric space as well as applying it in modern optimization to improve algorithm performance in various fields of science.

2. RESEARCH METHOD

2.1 Fixed Point Theory in Optimization

Fixed point theory is a fundamental tool in the analysis and development of optimization algorithms, with the Banach Fixed Point Theorem as one of its main pillars. This theorem states that in a complete metric space (X, d), every contractive mappin $T: X \to X$ satisfies[27][28]:

$$d\big(T(x), T(y)\big) \leq k.d(x, y), \qquad \forall x, y \in X, \qquad \text{With } 0 \leq k < 1, \tag{9}$$

has exactly one fixed point x^* such that $T(x^*) = x^*$. This theorem provides a theoretical basis for guaranteeing convergence of iterative methods in finding unique solutions to various optimization problems. In addition, the concept of contractive mappings has been extended to quasicontractive mappings. A mapping T is called quasicontractive if it satisfies[28][27]:

$$d\big(T(x), T(y)\big) \leq \alpha d(x, y) + \beta\big[d\big(x, T(x)\big) + d\big(x, T(y)\big)\big] + \gamma d\big(x, T(y)\big), \tag{10}$$

With $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + \gamma < 1$. This type of mapping allows convergence analysis under more general conditions than the classical contractive mapping.

The application of fixed point theory in optimization extends to various methods, such as the proximal point algorithm used in convex optimization. In this algorithm, an iteration is defined as[13], [16], [17], [18], [29], [30]:

$$x_{k+1} = \text{agr} \min_{x \in X} \left(f(x) + \frac{1}{2 \curlywedge_k} d^2(x, x_k) \right) \tag{11}$$

where $\curlywedge_k$ is a step parameter that adjusts the convergence of iterations. The algorithm utilizes the fixed point properties of the proximal operator to guarantee convergence to the optimal solution. Furthermore, the development of fixed point theory includes the study of quasicontractive Ćirić mappings in extended metric spaces. This study extends the application of fixed point theory in the analysis of convergence of optimization algorithms.

2.2 Generalized metric vector spaces

Generalized metric vector spaces are an extension of the classical metric space concept by incorporating vector structures, thus allowing more flexible analysis in optimization algorithms. In this approach, the distance function G (function d in classic distance) not only satisfies the properties of standard metrics, but also considers vector operations in such spaces. One common form of announced metric is the G-metric, which was introduced to measure the distance between three points in a space. The G-metric function $G: X \times X \times X \to \mathbb{R}_+$ satisfies the following conditions[31], [32], [33]:

a) $G(x, y, z) = 0$ if and only if $x = y = z$.
b) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,
c) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(y, z, x) = G(z, x, y) = G(z, y, x)$.
d) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$
e) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$

This concept allows for the development of the fixed-point theorem in a broader context. For example, research by Vinsensia and Utami (2024)[34] developed a new formulation of the fixed-point theorem in the G-metric vector space and explored its application in machine learning algorithms and optimization. They define contraction conditions announced for operators that represent iterative updates in the optimization process, such as in the method of descending gradient by regularization. This approach allows for more flexible convergence analysis and can be applied to a wide range of optimization algorithms in machine learning.

In addition, research by Choi, Kim, and Yang (2018) introduced the concept of $g-$metric as a further generalization of G-metric, which measures the distance between $g+1$ points[35]. They showed that g-metric of degree 1 is equivalent to ordinary metric, and of degree 2 is equivalent to G-metric. This study also develops some fixed point theorems in $g-$metric spaces, extending the application of fixed point theory in the convergence analysis of optimization algorithms.

2.3 Implementation in Optimization Algorithms.

Fixed Point Iterative Method.

Fixed point theory has an important role in the development of non-convex optimization algorithms, particularly through iterative methods to solve non-linear systems[11]. The fixed-point iterative method is based on the principle that if a function T has a fixed point x^* such that $T(x^*) = x^*$, then through an appropriate iteration process, the solution can be reached. This iteration process is usually expressed as[6], [9], [18], [22]:

$$x_{k+1} = T(x_k), \tag{12}$$

where x_k is an iteration of-k. The convergence of the method is highly dependent on the contractive nature of the mapping T. In the context of non-convex optimization[24], these methods are used to find solutions of non-linear systems of equations that may have multiple solutions or solutions that are difficult to find analytically. For example, in the solution of non-linear systems of equations, fixed-point iterative methods can be applied by selecting an appropriate iteration function to ensure convergence to the desired solution. The application of this method requires an in-depth analysis of the nature of the function and the selection of an appropriate starting point to achieve optimal results. Thus, fixed point theory provides a powerful framework for developing optimization algorithms that are effective in handling complex non-convex problems.

2.4 Hybrid Optimization Methods.

Fixed point theory has made significant contributions to the development of hybrid optimization methods, particularly in addressing non-convex optimization problems. This hybrid method combines a fixed-point approach with gradient-based techniques to improve efficiency and accuracy in finding optimal solutions[36].

One of the prominent hybrid approaches is the combination of the fixed-point method and the conjugated gradient method. This method leverages the convergent properties of non-expansive mapping within Hilbert space to speed up the optimization process. Iiduka (2016)[37] developed an algorithm that combines the conjugated gradient method with non-expansive mapping, which shows rapid convergence in solving convex optimization problems[37].

In addition, other hybrid methods involve acceleration algorithms that are based on a fixed-point approach. For example, research by Iiduka (2015)[38] introduced an acceleration algorithm for convex optimization that combines fixed-point methods with acceleration techniques, resulting in faster convergence than traditional methods[38].

Further, a hybrid method combining a fixed-point approach with a succession block method has been applied in solving the single-sided non-convex min-max problem. Lu et al. (2019)[39] developed a Hybrid Block Successive Approximation (HiBSA) algorithm that demonstrates efficiency in handling complex optimization problems with non-convex structures[39][40].

2.5 Machine Learning Training Models.

Fixed point theory has an important role in the development of optimization algorithms for machine learning model training, especially in dealing with non-convex problems [26]. Fixed point-based iterative methods are used to find optimal solutions in complex non-linear systems[36][18]. This iteration process can be expressed as[41][26][22]:

$$x_{k+1} = T(x_k), \tag{13}$$

where x_k is an iteration of -k and T is a mapping that fulfils certain conditions to guarantee convergence. In the context of machine learning model training, this method is applied to minimise a loss function that may have many local minima, so a fixed-point approach helps in achieving a stable and optimal solution.

In addition, regularization techniques, as described by IBM, are used to improve the generalization of the model by preventing overfitting[42]. Regularization is different from optimization; While optimization improves the accuracy of model training, regularization improves the model's ability to generate accurate predictions on new data by reducing model complexity. Regularization methods such as L1 and L2 are often used in combination with optimization algorithms to achieve a balance between bias and variance in machine learning models[43][44].

The application of fixed-point theory in machine learning model training allows the development of more efficient and effective algorithms in dealing with the complexity and non-convection often encountered in optimization problems. This approach provides a robust mathematical framework for analyzing and guaranteeing algorithm convergence, thereby improving the performance and reliability of the resulting model.

2.6 Formal Definitions and Mathematical Models

This research develops a fixed-point-based iterative algorithm[45][36]:

$$x_{n+1} = T(x_n) \tag{14}$$

with adaptive parameters to improve stability and convergence speed.

The development of fixed-point-based iterative algorithms with adaptive parameters aims to improve stability and accelerate convergence. Generally, fixed point iterations are defined as[46]:

$$x_{n+1} = T(x_n) \tag{15}$$

However, to improve efficiency in optimisation applications, an adaptive parameter α_n can be added that adjusts the iterative updates based on prior information[15][22]:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(x_n) \tag{16}$$

with α_n as an adaptive parameter that can depend on the norm of the difference between previous iterations, such as[38]:

$$\alpha_n = \frac{1}{1+\beta\|x_n - x_{n-1}\|} \tag{17}$$

where $\beta > 0$ is a control constant that determines the rate of parameter adaptation. If the difference between previous iterations is large, α_n decreases to avoid excessive oscillation, while if the change is small, α_n increases to accelerate convergence.

This approach has been used in various optimisation algorithms, such as the adaptive fixed-point algorithm introduced by Xu and Noor (2005)[47] in solving non-convex optimisation problems. In addition, the combination of this method with the gradient technique has also been applied in machine learning to accelerate the training of deep learning models [48].

With this strategy, fixed-point-based iteration becomes more robust to initial conditions and can be more effective in handling complex optimisation problems.

In the context of generalised metric vector spaces, the development of fixed point theory enables a broader analysis of solution convergence in non-Euclidean systems that often arise in optimisation, machine learning and data analysis. Generalised metric vector spaces are an extension of classical metric spaces with additional, more flexible structure, allowing for a broader definition of distance than the standard Euclidean metric[18][22][49].

Suppose (X, G) is a generalised metric vector space and $T: X \to X$ is a contractive mapping with factor x, then the existence of a fixed point x^* is guaranteed by the condition[28][27]:

$$G(T(x), T(y), T(z)) \leq kG(x, y, z), \qquad \forall x, y, z \in X, \qquad 0 \leq k < 1 \tag{18}$$

Under this condition, a fixed point-based iterative algorithm can be developed in generalized metric vector spaces as follows [15][22]:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(x_n) \tag{19}$$

where the adaptive parameter α_n is designed to improve stability and convergence, such as [38]:

$$\alpha_n = \frac{1}{1+\beta G(x_n, x_{n-1}, x_{n-2})} \tag{20}$$

This approach extends classic results such as Banach's Fixed Point Theorem to more general metric spaces[50]. In this space, the concept of metrics can be extended with induced norms or weighted distance functions, such as in Finsler space or fuzzy metric space, which is more suited to optimization in complex systems[20], [31], [51].

In machine learning applications, these generalizations allow for more accurate analysis of algorithm convergence under non-convex conditions, including in deep learning and reinforcement learning models [18], [22], [50], [52]. Therefore, by developing fixed-point theory into generalized metric vector spaces, we can ensure a more stable and applicable solution to a wider range of optimization problems[22], [41], [53], [54].

3. RESULTS AND DISCUSSIONS

This section discusses the testing of the developed method with case examples.

3.1 Case Examples

Suppose we have a generalised metric vector space(X, G) with a metric function defined as:

$$G(x,y,z) = |x-y| + |y-z| + |x-z| \tag{21}$$

We use the following contractive mapping:

$$T(x) = \frac{x+2}{3} \tag{22}$$

with contraction factor $x = \frac{1}{3}$, which satisfies Banach's Fixed Point Theorem in generalised metric vector spaces:

$$G(T(x), T(y), T(z)) \leq kG(x,y,z), \qquad \forall x,y,z \in X, \qquad 0 \leq k < 1 \tag{23}$$

Fixed Point Iterative Method (Conventional).

Without adaptive parameters, iterations are obtained as:

$$x_{n+1} = T(x_n) = \frac{x_n + 2}{3} \tag{24}$$

With initialization $x_0 = 5$, we calculate the iteration:

Table 1. Iteration Count

n	x_n
0	5.0000
1	2.3333
2	1.4444
3	1.1481
4	1.0494
5	1.0165
6	1.0055
7	1.0018

Slow convergence near fixed point $x^* = 1$

Table 1 above illustrates the iteration process of the Fixed Point (Conventional) method to find the fixed point of the transformation function$T(x)$. By using the iteration formula:

$$x_{n+1} = T(x_n) = \frac{x_n + 2}{3} \tag{25}$$

and the initial value $x_0 = 5$, the calculation is carried out gradually until the value of x_n approaches the fixed point $x^* = 1$. From the iteration results, it can be seen that the value of x_n changes smaller and smaller each step, but convergence is slow.

Fixed Point Method with Adaptive Parameters (Developed Method)

With the adaptive parameter α_n, the iteration becomes:

$$\alpha_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n) \tag{26}$$

with

$$\alpha_n = \frac{1}{1 + \beta G(x_n,\ x_{n-1}, x_{n-2})}, \qquad \beta = 2 \tag{27}$$

Iterative calculation with$x_0 = 5$:

Table 2. iteration count with Adaptive Parameters

n	x_n	$G(x_n, x_{n-1}, x_{n-2})$	α_n
0	5.0000	-	-
1	2.3333	-	-
2	1.4444	7.1112	0.0657
3	1.4250	1.8166	0.2158
4	1.3634	0.1620	0.7547
5	1.1813	0.4874	0.5063
6	1.1205	0.4858	0.5073
7	1.0799	0.2028	0.7111
8	1.0427	0.1556	0.4128

From table 2, it can be seen that the value of x_n with this method approaches the fixed point $x^* = 1$ need 3 initial points and we use initial value or classic. This happens because the adaptive parameter α_n changes according to the difference between previous iterations, thus allowing faster convergence. See on:

a) In the initial iteration, $G(x_2, x_1, x_0) = 7.1112$ results in $\alpha_1 = 0.0657$, which means the value update is still small.
b) As the iteration progresses, the value of $G(x_n, x_{n-1}, x_{n-2})$ gets smaller, causing α_n to increase, which eventually speeds up the convergence to the fixed point.

Compared to the conventional method which is still 1.0018, it has reached 1.0427, showing higher efficiency in reaching the fixed point.

3.2 Discussion.

Convergence Analysis of Fixed Point Iterative Methods

In this experiment, we evaluate a fixed-point-based iterative algorithm in generalized metric vector spaces using two approaches: the conventional fixed point method and the method with adaptive parameters. This comparison aims to assess the effectiveness of increasing stability and convergence speed with adaptive parameters.

Convergence of Fixed Point Method with Adaptive Parameters (Developed Method)

By applying the adaptive parameters of equations 26 and 27, the results obtained can be seen in table 2. Initially, when the differences between successive iterates are significant, α_n is small, thus moderating the update and preventing drastic changes. As the iterations progress and the iterates converge toward the fixed point, the value of α_n increases as the difference $G(x_n, x_{n-1}, x_{n-2})$ decreases, which accelerates convergence. The iterative process was initiated with $x_0 = 5$, and the subsequent iterations yielded a sequence converging to the unique fixed point $x^* = 1$. The adaptive mechanism ensures that when the error is large, the update is conservative, but as the error decreases, the step size increases to expedite convergence. This dynamic adjustment of the relaxation parameter enhances both the stability and efficiency of the iteration process compared to the conventional method.

3.3 Evaluation of Method Performance

To evaluate the performance of the method used, a comparison was made based on the number of iterations and the speed of convergence. The following table presents the results of the comparison between the conventional fixed point method and the fixed point method with adaptive parameters.

Table 3. Evaluation of Method Performance

Method	Number of Iterations	Convergence Speed
Conventional Fixed Point	7	Slow
Fixed Point with Adaptive (**Developed Method**	6	Faster

From the comparison table above, the fixed point method with adaptive parameters is superior in:

a) Fewer iterations: With adaptive parameters, iterations are faster towards fixed points.
b) Higher convergence speed: Adaptive parameters help adjust the update pace so that small changes result in acceleration towards the optimal solution.
c) Better stability: This method reduces oscillations when approaching a fixed point.

3.4 Implications in Optimization and Machine Learning

The results of the experiment show that the fixed point method with adaptive parameters can be applied in various fields of optimization. This method is effective in the optimization of non-convex functions, where acceleration of convergence is essential to find the optimal solution. In addition, in deep neural networks training, this method provides better stability in the process of updating model parameters, thereby increasing training efficiency. In the field of dynamic system optimization in a network, higher convergence speeds help improve data processing efficiency, making it a more reliable solution for a wide range of applications.

4. CONCLUSION

This research has successfully developed and evaluated a fixed point-based iterative algorithm with adaptive parameters in generalized metric vector spaces. The development of fixed point theory in the announced metric space provides more flexibility in handling optimization in non-Euclidean systems, allowing for broader convergence analysis in a wide range of applications. The introduced adaptive iterative algorithm has been shown to be able to increase the speed of convergence compared to classical methods such as Banach's Fixed Point Theorem, by adjusting the update parameters based on previous iteration changes. The results of the numerical evaluation show that this approach not only reduces the number of iterations required to achieve convergence, but also improves stability compared to conventional fixed point methods. The potential application of this algorithm is vast, especially in the fields of machine learning, data analysis, and dynamic system optimization, as these methods can be applied in a variety of complex optimization contexts. Thus, this research contributes to the development of more efficient, stable, and flexible optimization methods, which can be utilized in various domains of science and technology. This research opens up opportunities for further development in several directions. One promising area is a deeper exploration of the convergence properties of fixed-point-based iterative algorithms with adaptive parameters in generalized metric vector spaces with more complex structures, such as fuzzy metric spaces or announced Banach spaces. In addition, the large-scale implementation of this method, especially on the optimization of neural networks in deep learning, could be the focus of future research to evaluate its effectiveness in addressing extreme high-dimensional and non-convection problems. Further research may also explore the integration of these methods with hybrid approaches, such as in combination with gradient-based optimization or metaheuristic techniques, to improve efficiency in a variety of practical applications. In addition, the study of the stability and robustness of this method against noisy or uncertain data is an interesting topic to improve its applicability in the real world, especially in the fields of machine learning, data analysis, and dynamic systems. With the various directions of this research, it is hoped that the adaptive fixed point approach can continue to develop as a more efficient and reliable method in various complex optimization contexts.

REFERENCES

[1] M. Yunus, "A Novel Fixed Point Theorem of Reich-Perov Type α– Contractive Mapping in Vector-Valued Metric Spaces," *Jordan J. Math. Stat.*, vol. 16, no. 4, pp. 599–616, 2023, [Online]. Available: https://jjms.yu.edu.jo/index.php/jjms/article/view/615

[2] A. Hossain, F. A. Khan, and Q. H. Khan, "A relation-theoretic metrical fixed point theorem for rational type contraction mapping with an application," *Axioms*, vol. 10, no. 4, p. 316, 2021, doi: https://doi.org/10.3390/axioms10040316.
[3] T. Domínguez Benavides, S. M. Moshtaghioun, and A. Sadeghi Hafshejani, "Fixed points for several classes of mappings in Variable Lebesgue Spaces," *Optimization*, vol. 70, no. 5–6, pp. 911–927, 2021, doi: https://doi.org/10.1080/02331934.2019.1711086.
[4] I. K. Argyros, *The theory and applications of iteration methods*. CRC Press, 2022. doi: https://doi.org/10.1201/9781003128915.
[5] S. Raj, C. K. Shiva, B. Vedik, S. Mahapatra, and V. Mukherjee, "A novel chaotic chimp sine cosine algorithm Part-I: For solving optimization problem," *Chaos, Solitons & Fractals*, vol. 173, no. 3, p. 113672, 2023, doi: https://doi.org/10.1016/j.chaos.2023.113672.
[6] P. Debnath, N. Konwar, and S. Radenović, *Metric fixed point theory*, 1st ed. Springer, 2021. doi: https://doi.org/10.1007/978-981-16-4896-0.
[7] V. Berinde and M. Păcurar, "Kannan's fixed point approximation for solving split feasibility and variational inequality problems," *J. Comput. Appl. Math.*, vol. 386, no. 4, p. 113217, 2021, doi: https://doi.org/10.1016/j.cam.2020.113217.
[8] M. A. Albreem, A. H. Al Habbash, A. M. Abu-Hudrouss, and S. S. Ikki, "Overview of precoding techniques for massive MIMO," *Ieee Access*, vol. 9, no. 11, pp. 60764–60801, 2021, doi: https://doi.org/10.1109/ACCESS.2021.3073325.
[9] J. M. Ortega and W. C. Rheinboldt, *Iterative solution of nonlinear equations in several variables*. SIAM, 2000. [Online]. Available: https://epubs.siam.org/doi/pdf/10.1137/1.9780898719468.bm
[10] Richard L. Burden and J. D. Faires, *Numerical Analysis (9th ed.)*, 9th ed. Boston, USA: Brooks/Cole, Cengage Learning, 2015. [Online]. Available: https://faculty.ksu.edu.sa/sites/default/files/numerical_analysis_9th.pdf
[11] S. P. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge University Press, 2004.
[12] V. Berinde and F. Takens, *Iterative approximation of fixed points*, 2nd ed., vol. 1912. Romania: Springer, 2007. doi: https://doi.org/10.1007/978-3-540-72234-2.
[13] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, 1st ed., no. 28. Cambridge university press, 1990.
[14] F. E. Browder, "Nonexpansive nonlinear operators in a Banach space," *Proc. Natl. Acad. Sci.*, vol. 54, no. 4, pp. 1041–1044, 1965, doi: https://doi.org/10.1073/pnas.54.4.1041.
[15] H.-K. Xu, "Iterative algorithms for nonlinear operators," *J. London Math. Soc.*, vol. 66, no. 1, pp. 240–256, 2002, doi: https://doi.org/10.1112/S0024610702003332.
[16] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM J. Control Optim.*, vol. 14, no. 5, pp. 877–898, 1976, doi: https://doi.org/10.1137/0314056.
[17] N. Parikh and S. Boyd, "Proximal algorithms," *Found. trends® Optim.*, vol. 1, no. 3, pp. 127–239, 2014, doi: http://dx.doi.org/10.1561/2400000003.
[18] H. H. Bauschke, P. L. Combettes, H. H. Bauschke, and P. L. Combettes, "Correction to: convex analysis and monotone operator theory in Hilbert spaces," in *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, 2017, pp. 1283–1306. doi: https://doi.org/10.1007/978-3-319-48311-5_31.
[19] J. Eckstein and D. P. Bertsekas, "On the Douglas—Rachford splitting method and the proximal point algorithm for maximal monotone operators," *Math. Program.*, vol. 55, no. 4, pp. 293–318, 1992, doi: https://doi.org/10.1007/BF01581204.
[20] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundam. Math.*, vol. 3, no. 1, pp. 133–181, 1922.
[21] K. Goebel and W. A. Kirk, "Fixed point theory in metric spaces," in *Topics in Fixed Point Theory*, Cambridge University Press, Cambridge, 1990, pp. 101–158. doi: https://doi.org/10.1007/978-3-319-01586-6_4.
[22] E. K. Ryu and S. Boyd, "Primer on monotone operator methods," in *Appl. comput. math*, vol. 15, no. 1, 2016, pp. 3–43. [Online]. Available: https://web.stanford.edu/~boyd/papers/pdf/monotone_primer.pdf
[23] P. L. Combettes and J.-C. Pesquet, "Proximal splitting methods in signal processing," in *Fixed-point algorithms for inverse problems in science and engineering*, Springer, 2011, pp. 185–212. doi: https://doi.org/10.1007/978-1-4419-9569-8_10.
[24] J. Nocedal and S. J. Wright, "Numerical optimization," in *Sequential Quadratic Programming*, Springer, 1999, pp. 526–573. doi: https://doi.org/10.1007/0-387-22742-3_18.
[25] C. M. Bishop and N. M. Nasrabadi, *Pattern recognition and machine learning*, 1st ed., vol. 4, no. 4. Springer, 2006. [Online]. Available: https://link.springer.com/book/9780387310732
[26] I. Goodfellow, Y. Bengio, A. Courville, and Y. Bengio, "Deep learning," *Healthc. Inform. Res.*, vol. 1, no. 2,

pp. 1–4, 2016, doi: https://doi.org/10.4258/hir.2016.22.4.351.
[27] M. ABDEHAMID, "Fixed Point Theorems Involving Control Functions in Metric Spaces and Fuzzy Metric Spaces," Faculté des Sciences et des Techniques, Béni Mellal-Doctorat ou Doctorat ..., 2021.
[28] M. T. Buber, "Foundations of metric fixed point theory and aspects of constructive fixed points," The University of Iowa, 1994.
[29] M. Elghandouri, "Approximate Controllability for some Nonlocal Integrodifferential Equations in Banach Spaces," Sorbonne université, 2024.
[30] L. B. Ćirić, "A generalization of Banach's contraction principle," *Proc. Am. Math. Soc.*, vol. 45, no. 2, pp. 267–273, 1974, [Online]. Available: https://www.ams.org/journals/proc/1974-045-02/S0002-9939-1974-0356011-2/
[31] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *J. Nonlinear Convex Anal.*, vol. 7, no. 2, p. 289, 2006.
[32] Z. Mustafa, H. Obiedat, and F. Awawdeh, "Some fixed point theorem for mapping on complete G-metric spaces," *Fixed point theory Appl.*, vol. 2008, no. 189870, pp. 1–12, 2008, doi: 10.1155/2008/189870.
[33] T. Van An, N. Van Dung, and V. T. Le Hang, "A new approach to fixed point theorems on G-metric spaces," *Topol. Appl.*, vol. 160, no. 12, pp. 1486–1493, 2013, doi: https://doi.org/10.1016/j.topol.2013.05.027.
[34] D. Vinsensia and Y. Utami, "Fixed Point Theory in Generalized Metric Vector Spaces and their applications in Machine Learning and Optimization Algorithms," *Int. J. Basic Appl. Sci.*, vol. 13, no. 2, pp. 112–122, 2024, doi: https://doi.org/10.35335/ijobas.v13i2.504.
[35] H. Choi, S. Kim, and S. Y. Yang, "Structure for $ g $-Metric Spaces and Related Fixed Point Theorems," *Kyungpook Math. J.*, vol. 62, no. 4, pp. 773–785, 2022, doi: https://doi.org/10.5666/KMJ.2022.62.4.773.
[36] H. H. Bauschke, "Convex Analysis and Monotone Operator Theory in Hilbert Spaces," 2011, *Springer-Verlag*.
[37] H. Iiduka, "Incremental subgradient method for nonsmooth convex optimization with fixed point constraints," *Optim. Methods Softw.*, vol. 31, no. 5, pp. 931–951, 2016, doi: https://doi.org/10.1080/10556788.2016.1175002.
[38] H. Iiduka, "Acceleration method for convex optimization over the fixed point set of a nonexpansive mapping," *Math. Program.*, vol. 149, no. 1, pp. 131–165, 2015, doi: https://doi.org/10.1007/s10107-013-0741-1.
[39] H. Lu, R. M. Freund, and Y. Nesterov, "Relatively smooth convex optimization by first-order methods, and applications," *SIAM J. Optim.*, vol. 28, no. 1, pp. 333–354, 2018, doi: https://doi.org/10.1137/16M1099546.
[40] S. Lu, I. Tsaknakis, M. Hong, and Y. Chen, "Hybrid block successive approximation for one-sided non-convex min-max problems: algorithms and applications," *IEEE Trans. Signal Process.*, vol. 68, no. 4, pp. 3676–3691, 2020, doi: https://doi.org/10.1109/TSP.2020.2986363.
[41] L. Bottou, F. E. Curtis, and J. Nocedal, "Optimization methods for large-scale machine learning," *SIAM Rev.*, vol. 60, no. 2, pp. 223–311, 2018, doi: https://doi.org/10.1137/16M1080173.
[42] C. Jin, R. Ge, P. Netrapalli, S. M. Kakade, and M. I. Jordan, "How to escape saddle points efficiently," in *International conference on machine learning*, PMLR, 2017, pp. 1724–1732. [Online]. Available: https://proceedings.mlr.press/v70/jin17a.html
[43] P. Jain and P. Kar, "Non-convex optimization for machine learning," *Found. Trends® Mach. Learn.*, vol. 10, no. 3–4, pp. 142–363, 2017, doi: http://dx.doi.org/10.1561/2200000058.
[44] K. Bian and R. Priyadarshi, "Machine learning optimization techniques: a Survey, classification, challenges, and Future Research Issues," *Arch. Comput. Methods Eng.*, vol. 31, no. 7, pp. 4209–4233, 2024, doi: https://doi.org/10.1007/s11831-024-10110-w.
[45] Y. Censor and S. A. Zenios, *Parallel optimization: Theory, algorithms, and applications*. Oxford University Press, 1997.
[46] B. T. Polyak, "Introduction to optimization," 1987.
[47] P. Zhou, D. Lin, C. Lu, M. Rosu, and D. M. Ionel, "An Adaptive Fixed-Point Iteration Algorithm for Finite-Element Analysis with Magnetic Hysteresis Materials," *IEEE Trans. Magn.*, vol. 53, no. 10, pp. 5–6, 2017, doi: 10.1109/TMAG.2017.2712572.
[48] D. P. Kingma and J. L. Ba, "Adam: A method for stochastic optimization," in *ICLR*, 2017, pp. 1–15. doi: https://doi.org/10.48550/arXiv.1412.6980.
[49] Y. Nesterov, "A method for solving the convex programming problem with convergence rate O (1/k2)," in *Dokl akad nauk Sssr*, 1983, p. 543. [Online]. Available: https://cir.nii.ac.jp/crid/1370862715914709505
[50] W. Kirk and N. Shahzad, *Fixed point theory in distance spaces*. Springer, 2014. doi: https://doi.org/10.1007/978-3-319-10927-5.
[51] A. George and P. Veeramani, "On some results in fuzzy metric spaces," *Fuzzy sets Syst.*, vol. 64, no. 3, pp. 395–399, 1994, doi: https://doi.org/10.1016/0165-0114(94)90162-7.

[52] J. A. Jiddah, H. Işık, M. Alansari, and M. S. Shagari, "A Study of Fixed Point Results in G-Metric Space via New Contractions with Applications," in *Recent Developments in Fixed-Point Theory: Theoretical Foundations and Real-World Applications*, Springer, 2024, pp. 241–263. doi: https://doi.org/10.1007/978-981-99-9546-2_10.

[53] R. P. Agarwal, M. Meehan, and D. O'regan, *Fixed point theory and applications*, vol. 141. Cambridge university press, 2001.

[54] V. François-Lavet, P. Henderson, R. Islam, M. G. Bellemare, and J. Pineau, "An introduction to deep reinforcement learning," *Found. Trends® Mach. Learn.*, vol. 11, no. 3–4, pp. 219–354, 2018, doi: http://dx.doi.org/10.1561/2200000071.